SD

MA 214
3/27/2023
Quiz 5
Version A

Full name: ______________________

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1. Use the table of Laplace transforms to take the Laplace transform of both sides of this IVP, letting $\mathcal{L}\{y\} = Y$. Then solve for Y. Be sure to use the given initial conditions. **<u>You do not need to find the inverse Laplace transform of Y once you've solved for it</u>**.

$$2y'' + y' + y = e^{3t}, \qquad y(0) = 3, \qquad y'(0) = 1$$

$$\mathcal{L}\{2y''+y'+y\} = \mathcal{L}\{e^{3t}\}$$

$$\Rightarrow 2\mathcal{L}\{y''\} + \mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{e^{3t}\}$$

$$\Rightarrow 2\left(s^2Y(s) - sy(0) - y'(0)\right) + \left(sY(s) - y(0)\right) + Y(s) = \frac{1}{s-3}$$

$$\Rightarrow 2\left(s^2Y(s) - 3s - 1\right) + \left(sY(s) - 3\right) + Y(s) = \frac{1}{s-3}$$

$$\Rightarrow 2s^2Y(s) - 6s - 2 + sY(s) - 3 + Y(s) = \frac{1}{s-3}$$

$$\Rightarrow (2s^2 + s + 1)Y(s) = \frac{1}{s-3} + 6s + 5 \Rightarrow Y(s) = \frac{\frac{1}{s-3} + 6s + 5}{2s^2 + s + 1}$$

2. Determine the inverse Laplace transform of the function using the table of Laplace transforms.

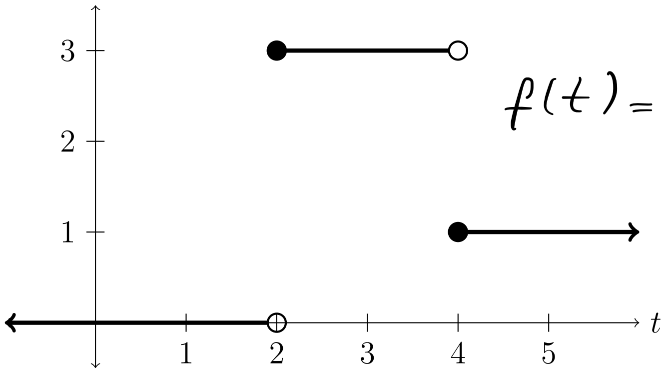
$$F(s) = \frac{2}{s^2} + \frac{3s}{s^2+10} + \frac{2}{s+5}$$

$$f(t) = \mathcal{L}^{-1}\{F\}(t) = \mathcal{L}^{-1}\left\{\frac{2}{s^2} + \frac{3s}{s^2+10} + \frac{2}{s+5}\right\}(t)$$

$$= 2\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + 3\mathcal{L}^{-1}\left\{\frac{s}{s^2+10}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\}$$

$$= 2t + 3\cos(\sqrt{10}\,t) + 2e^{-5t}$$

3. Write an equation for the following function, using Heaviside notation. (You may use either $u(t)$ or $h(t)$ to represent the Heaviside function.)



$$f(t) = \begin{cases} 0, & \text{if } t < 2 \\ 3, & \text{if } 2 \leq t < 4 \\ 1, & \text{if } t \geq 4 \end{cases}$$

$$f(t) = 3\left(u_2(t) - u_4(t)\right) + 1 \cdot u_4(t)$$

$$= 3u_2(t) - 3u_4(t) + u_4(t) = 3u_2(t) - 2u_4(t)$$

4. Use the integral definition of the Laplace transform to determine the Laplace transform of $f(t) = t$. You must show every step of your work.

$$\mathcal{L}\{t\}(s) = \int_0^\infty e^{-st}\, t\, dt = \lim_{N\to\infty} \int_0^N e^{-st} dt$$

Integration by parts: $\int_0^\infty u\,dv = uv\Big|_0^\infty - \int_0^\infty v\,du$

Let $u = t \Rightarrow \frac{du}{dt} = 1 \Rightarrow du = dt$

$$dv = e^{-st} dt \Rightarrow \frac{dv}{dt} = e^{-st} \Rightarrow v = \frac{e^{-st}}{-s}$$

$$\int_0^\infty e^{-st}\, t\, dt = \int_0^\infty u\,dv = uv\Big|_{t=0}^\infty - \int_0^\infty v\,du$$

$$= -t\frac{e^{-st}}{s}\Bigg|_{t=0}^\infty - \int_0^\infty -\frac{e^{-st}}{s}\, dt$$

$$= \lim_{N\to\infty}\left[-t\cdot\frac{e^{-st}}{s}\Bigg|_{t=0}^N + \frac{1}{s}\int_0^N e^{-st} dt\right]$$

$$= \lim_{N\to\infty} \left[-\left(\underbrace{\cancel{\frac{N}{e^{SN}}}}_{\to 0 \text{ as } N\to\infty} \cdot \frac{1}{s} - \underbrace{\cancel{\frac{0 \cdot e^{0}}{s}}}_{=0} \right) + \frac{1}{s} \cdot \frac{e^{-st}}{-s} \Big|_0^N \right]$$

Note: $\lim_{N\to\infty} \frac{N}{e^{sN}} = 0$ for $s>0$ since e^{sN} growths faster than N.

$$= \lim_{N\to\infty} -\frac{1}{s^2}\left[e^{-sN} - e^{0}\right] = \lim_{N\to\infty} \frac{1}{s^2} \cdot \left[1 - \underbrace{\cancel{e^{-sN}}}_{\to 0 \text{ as } N\to\infty}\right]$$

$$= \frac{1}{s^2}$$

b/c $\lim_{N\to\infty} e^{-sN} = 0$ for $s>0$.

Thus, $\mathcal{L}\{t\} = \frac{1}{s^2}$

See the table for any $n = 1, 2, 3, \ldots$

$$\mathcal{L}\{t^n\}(s) = \frac{n!}{s^{n+1}}$$

For $n=1$: $\mathcal{L}\{t\} = \frac{1!}{s^{1+1}} = \frac{1}{s^2}$